

## Holographic $U(1)_A$ and string creation

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**ABSTRACT:** We analyze the resolution of the  $U(1)_A$  problem in the Sakai-Sugimoto holographic dual of large  $N_c$  QCD at finite temperature. It has been shown that in the confining phase the axial symmetry is broken at order  $1/N_c$ , in agreement with the ideas of Witten and Veneziano. We show that in the deconfined phase the axial symmetry remains unbroken to all orders in  $1/N_c$ . In this case the breaking is due to instantons which are described by spacelike D0-branes, in agreement with 't Hooft's resolution. The holographic dual of the symmetry breaking fermion condensate is a state of spacelike strings between the D0-brane and the flavor D8-branes, which result from a spacelike version of the string creation effect. In the intermediate phase of deconfinement with broken chiral symmetry the instanton gas approximation is possibly regulated in the IR, which would imply an  $\eta'$  mass-squared of order  $\exp(-N_c)$ .

**KEYWORDS:** D-branes, Intersecting branes models, Gauge-gravity correspondence.

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## 1. Introduction/history

QCD has three light flavors of quarks and therefore an approximate global  $SU(3)_R \times SU(3)_L$  chiral symmetry. In nature this symmetry is spontaneously broken to the diagonal subgroup  $SU(3)_V$ , and the corresponding pseudo-Goldstone bosons are identified with the eight light pseudo-scalar mesons - the  $\pi$ 's,  $K$ 's and  $\eta$ . The symmetry of the action is more precisely  $U(3)_R \times U(3)_L$ , and the unbroken part is given by  $SU(3)_V \times U(1)_V$ , where  $U(1)_V$  corresponds to Baryon number. This would seem to imply another pseudo-Goldstone boson for the broken axial symmetry  $U(1)_A$ , which is not observed. The candidate flavor-singlet  $\eta'$  meson is too massive. This is known as the  $U(1)_A$  problem.

A resolution to this problem was first given by 't Hooft [1]. The axial symmetry is broken by the anomaly, which is in turn related to instantons. Due to the presence of fermionic zero-modes in the instanton background one needs to include fermions in the path integral in order to get a non-vanishing amplitude. The correct combination is  $\det(\bar{\psi}_R \psi_L)$ , where the determinant is on the flavor indices, and the color indices are contracted in each bi-linear. This is invariant under the chiral symmetry  $SU(N_f)_R \times SU(N_f)_L$ , but not under the axial symmetry  $U(1)_A$ . The instanton-induced condensate  $\langle \det(\bar{\psi}_R \psi_L) \rangle$  therefore breaks  $U(1)_A$  non-perturbatively. Summing over instantons and anti-instantons in a dilute gas approximation then gives a new effective vertex in the action, which leads to an  $\eta'$  mass of order  $\exp(-8\pi^2/g_{\text{YM}}^2)$ .

However there are a couple of problems with the instanton picture. The first is that the dilute instanton gas approximation breaks down when the instantons become too large, and there is an IR divergence in the integral over the instanton size. The second problem is that the instanton picture appears to be in conflict with perturbative contributions to the  $\eta'$  mass in the quark model coming from quark-antiquark annihilation into gluons [2]. This conflict was made sharper in the large  $N_c$  limit by Witten [3]. Perturbative effects lead to an  $\eta'$  mass-squared of order  $1/N_c$  at large  $N_c$ , whereas the instanton gas picture would give a mass-squared of order  $\exp(-N_c)$ . Witten argued that the conflict is resolved by confinement, since in a confined gauge theory topological charge is not quantized, and it therefore does not make sense to think about instantons, which are quanta of topological charge. This lack of quantization is related to strong fluctuations of the topological charge density in the confining vacuum, which is measured by the topological susceptibility of the pure Yang-Mills theory  $\chi_g = (d^2 E/d\theta^2)_{\theta=0}$ . Witten and Veneziano derived the relation between this quantity and the  $\eta'$  mass [4, 5]

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2} \chi_g, \tag{1.1}$$

where  $f_\pi$  is the pion decay constant.<sup>1</sup> Since  $f_\pi^2 \sim \mathcal{O}(N_c)$  one would get an order  $1/N_c$  mass-squared if  $\chi_g \sim \mathcal{O}(1)$ . Witten and Veneziano argued that although  $\chi_g$  vanishes order by order in perturbation theory, it could be finite in the large  $N_c$  limit.

This proposal was confirmed using gauge/gravity duality. In [6] Witten proposed a gravity dual for four-dimensional pure Yang-Mills theory in terms of 4-branes in Type IIA string theory which are wrapped on a circle with anti-periodic fermions. Using the duality he then showed that  $\chi_g \sim \mathcal{O}(1)$  [7]. More recently Sakai and Sugimoto proposed a model for massless QCD by adding 8-branes and anti-8-branes to Witten's model [8]. The 8-branes and anti-8-branes provide the right- and left-handed quarks. This model exhibits chiral symmetry breaking geometrically by the connection of the 8-branes to the anti-8-branes. The mesons correspond to modes of the 8-branes in the connected configuration, and one can compute their spectrum from the geometry. In particular the  $\eta'$  mass was shown to satisfy precisely (1.1) with the  $\chi_g$  computed in [7].<sup>2</sup>

Witten's model for pure Yang-Mills theory and Sakai and Sugimoto's extension to massless QCD have also been analyzed at finite temperature [11]. It was shown that the model undergoes deconfinement and chiral-symmetry-restoration transitions. One of the interesting features of this model is that for some range of the parameters chiral symmetry restoration occurs at a higher temperature than deconfinement (otherwise they happen at the same temperature). In other words there is an intermediate phase where gluons are freed but quarks can still form stable mesonic bound states. In the deconfined phase one expects instantons to become relevant for  $U(1)_A$  breaking, and in particular to contribute to the  $\eta'$  mass in the intermediate phase.

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<sup>1</sup>We are using Veneziano's conventions [5], in which the pion decay constant is defined by  $\langle 0|\bar{q}\gamma^\mu\gamma^5\tau^a q|\pi^a(p)\rangle = if_\pi p^\mu\delta^{ab}$  with  $\text{Tr}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$ , which gives  $\langle 0|\bar{q}\gamma^\mu\gamma^5 q|\eta'(p)\rangle = i\sqrt{2N_f}f_\pi p^\mu$ . In real QCD this corresponds to  $f_\pi \approx 93$  MeV. In [4] the  $\sqrt{2}$  is absent, and this results in a factor 4 rather than 2 to in the expression for  $m_{\eta'}^2$ .

<sup>2</sup>For a description of  $U(1)_A$ -breaking in other gauge/gravity dual models see [9, 10].

The main goal of this paper is to understand whether and how 't Hooft's resolution is realized in the dual supergravity picture. Using the supergravity description we will show that in the deconfined phase  $\chi_g$  vanishes to order 1 at large  $N_c$ , and  $m_{\eta'}^2$  vanishes to order  $1/N_c$ . Furthermore, since these results are based on a topological argument, we will argue that they hold to all orders in the  $1/N_c$  expansion. The only contribution to  $U(1)_A$  breaking in this phase must therefore come from instantons. We will show that instanton charge, which corresponds in the supergravity picture to spacelike Euclidean 0-branes, is indeed quantized in this phase. The presence of both the 0-brane and the 8-branes in the background leads to a spacelike version of the familiar string-creation effect. We will show that the resulting spacelike strings correspond precisely to the  $U(1)_A$ -breaking condensate  $\langle \det(\bar{\psi}_R \psi_L) \rangle$ . We will also argue that the IR problem in the instanton sum for computing the mass of the  $\eta'$  may be resolved by a cutoff at the chiral-symmetry breaking scale. This implies an  $\exp(-N_c)$  behavior for the  $\eta'$  mass in the intermediate phase.

We begin in section 2 by briefly reviewing the models of Witten and Sakai-Sugimoto at zero and finite temperature, with emphasis on the  $U(1)_A$  issue. In section 3 we show that  $\chi_g = 0$  to order 1 in the deconfined phase, and  $m_{\eta'} = 0$  to order  $1/N_c$  in the intermediate phase of deconfinement with chiral symmetry breaking, and argue that these hold to all orders in  $1/N_c$ . In section 4 we discuss instantons in the dual supergravity picture, and show how they break  $U(1)_A$  geometrically by a spacelike version of the string creation effect. We have also included an appendix containing a worldsheet argument for the spacelike string.

## 2. Review of previous work

### 2.1 Holographic Yang-Mills

The gravity dual of four-dimensional  $SU(N_c)$  YM theory is constructed by wrapping  $N_c$  Type IIA D4-branes on a circle  $x_4 \sim x_4 + \beta_4$  with anti-periodic boundary conditions for the fermions, and taking the near-horizon limit [6]. The background is given by [12]

$$\begin{aligned}
 ds^2 &= \left(\frac{u}{R}\right)^{3/2} (-dt^2 + \delta_{ij} dx^i dx^j + f(u) dx_4^2) + \left(\frac{u}{R}\right)^{-3/2} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right) \\
 e^\Phi &= \left(\frac{u}{R}\right)^{3/4}, \quad F_4 = \frac{3N_c}{4\pi} \epsilon_4, \quad f(u) = 1 - \frac{u_\Lambda^3}{u^3},
 \end{aligned}
 \tag{2.1}$$

where  $R = (\pi g_s N_c)^{1/3} l_s$ , and  $u_\Lambda$  is related to the periodicity  $\beta_4$  of  $x_4$  as

$$\beta_4 = \frac{4\pi}{3} \left(\frac{R^3}{u_\Lambda}\right)^{1/2}.
 \tag{2.2}$$

The four-dimensional YM coupling is given by  $g_{YM}^2 = 2(2\pi)^2 g_s l_s / \beta_4$  [13], and the 't Hooft coupling is defined by  $\lambda \equiv g_{YM}^2 N_c$ . The topology of the space is  $\mathbb{R}^{3,1} \times D \times S^4$ . In particular the physical size of the circle goes to zero smoothly as  $u$  approaches  $u_\Lambda$  from above. This background gives confinement in the dual gauge theory, which can be seen by computing the quark-antiquark potential, corresponding to the action of a string with endpoints on the boundary at  $u \rightarrow \infty$  [14]. This can also be seen as a consequence of the non-contractibility

of the Polyakov loop in the Euclidean finite temperature background. The role of the gauge theory parameter  $\theta$  is played in the dual background by the RR 1-form [7]

$$\theta = \int_{S^1} C_1, \tag{2.3}$$

where  $S^1$  is the boundary at  $u \rightarrow \infty$  of the disk  $D$ . By Stokes' theorem it is therefore also related to the RR field strength  $F_2 = dC_1$  as

$$\int_D F_2 = \theta. \tag{2.4}$$

A non-trivial vacuum angle  $\theta$  therefore requires a non-trivial RR field strength. The solution to the equation of motion for  $C_1$  with this condition for the flux on the disk is given by

$$F_2 = \frac{c}{u^4} \theta du \wedge dx_4, \tag{2.5}$$

where  $c = 3u_\Lambda^3/\beta_4$ . Plugging this back into the kinetic term for  $C_1$  in the supergravity action and integrating over  $u, x_4$  and the  $S^4$  gives the four-dimensional energy density<sup>3</sup>

$$E(\theta) = \frac{4\pi}{3} \frac{c}{(2\pi l_s)^6} \theta^2 = \frac{1}{2} \chi_g \theta^2. \tag{2.6}$$

This confirms that  $\chi_g \sim \mathcal{O}(1)$  at large  $N_c$ .

## 2.2 The Sakai-Sugimoto model

Massless quarks are included by adding  $N_f$  8-branes and  $N_f$  anti-8-branes which are localized in  $x_4$  [8].<sup>4</sup> For  $N_f \ll N_c$  we regard the 8-branes as probes, and ignore their backreaction on the background.<sup>5</sup> This gives a  $U(N_f)_R \times U(N_f)_L$  symmetry, which is global from the 4-brane point of view, and new states transforming in the fundamental representations the gauge symmetry and the global symmetry from the 4-8 and 4- $\bar{8}$  strings. Since the 4-branes are extended along  $(x_1, \dots, x_4)$  the 4-8 strings have  $ND = 6$  directions with mixed boundary conditions. The NS sector of these strings is massive, and the R sector has a zero-energy ground state with (complex) degeneracy  $2^{4/2} = 4$ , corresponding to a massless Dirac fermion in the four-dimensional intersection of the branes. The GSO projection leaves a Weyl fermion of one chirality. For the 4- $\bar{8}$  strings the GSO projection is reversed, giving the other chirality. The low energy states are therefore massless

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<sup>3</sup>The actual result is  $E(\theta) \propto \min_k (\theta + 2\pi k)^2$ , where  $k$  is an integer. This is because  $\theta$  is an angle variable, whereas  $F_2$  and  $E$  are real numbers.

<sup>4</sup>For other ways of adding flavor see [15–17].

<sup>5</sup>It's not clear whether D8-branes can truly be treated in the probe approximation, since their backreaction on the background is significant. In particular the dilaton behaves linearly in the coordinate transverse to the 8-branes, which makes it difficult to compactify this coordinate. This is true even in the case with an equal number of 8-branes and anti-8-branes. Although the RR tadpole cancels, the dilaton tadpole actually adds.

right-handed fermions transforming as  $(\mathbf{N}_c, \mathbf{N}_f, \mathbf{1})$  and massless left-handed fermions transforming as  $(\mathbf{N}_c, \mathbf{1}, \mathbf{N}_f)$ . Since the 8-branes fill all the directions transverse to the 4-branes there is no obvious way of giving an explicit mass to the fermions.

The most striking feature of this model is that it exhibits chiral symmetry breaking in a beautifully simple and intuitive way. In the decoupling limit the 4-branes are replaced with their near-horizon background (2.1). Since the radial coordinate  $u$  does not extend down to the origin, the 8-branes and anti-8-branes must connect at some  $u = u_0$  (figure (1a)). The chiral symmetry of the 8-branes and anti-8-branes is therefore broken to the diagonal  $U(N_f)_V$ , and  $u_0$  defines the scale of chiral symmetry breaking. The connected configuration is a solution of the 8-brane DBI action in this background corresponding to a U-shaped curve  $\gamma(u, x_4) = 0$  in the  $(u, x_4)$  plane [8, 11]. The minimal value of  $u$  on the 8-branes  $u_0$  is related to the asymptotic separation  $L$  between the 8-branes and anti-8-branes. At maximal (antipodal) separation  $L = \beta_4/2$  the 8-branes connect at the minimal value of the background  $u_0 = u_\Lambda$ , and for small  $L$  ( $L \ll \beta_4$ )  $u_0 \propto R^3/L^2$ .

Mesons are described by modes of the 8-branes in the connected configuration. In particular the pseudo-scalar mesons which correspond to the Goldstone bosons of the broken chiral symmetry are identified with the 0-modes of the  $u$  component of the 8-brane worldvolume gauge field  $A_u$  (we are using a parameterization in which  $u$  is a worldvolume coordinate). The  $\eta'$  is in turn associated with  $\text{tr}(A_u)$ . In the DBI action all of these modes, including  $\eta'$ , are massless. However  $\eta'$  becomes massive due to an anomaly in the bulk. The gauge invariant RR field strength is shifted by the presence of the 8-branes to

$$\tilde{F}_2 = dC_1 + i \text{tr}(A_u) \delta(\gamma(u, x_4)) du \wedge dx_4. \tag{2.7}$$

Thus under a worldvolume gauge transformation the RR 1-form changes by

$$\delta_\Lambda C_1 = -i \text{tr}(\Lambda) \delta(\gamma(u, x_4)) dx_4. \tag{2.8}$$

At  $u \rightarrow \infty$  the 8-brane embedding curve  $\gamma(u, x_4)$  reduces to  $x_4 = \mp L/2$ , corresponding to the asymptotic positions of the 8-branes and anti-8-branes, respectively. It follows that  $\theta$  transforms as

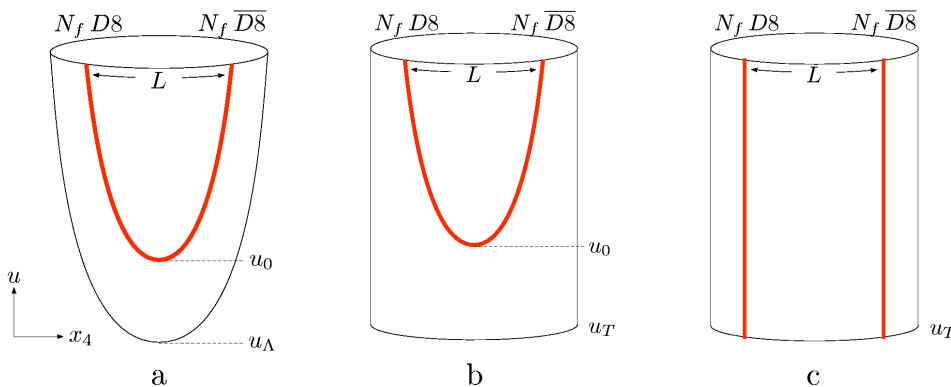
$$\delta_\Lambda \theta = -i \text{tr}(\Lambda|_{x_4=-L/2} - \Lambda|_{x_4=L/2}). \tag{2.9}$$

The r.h.s. is the gauge transformation parameter in the relative overall  $U(1)$  of the 8-branes and anti-8-branes, which is precisely  $U(1)_A$ , and this is the correct transformation of  $\theta$  under the axial symmetry. Integrating (2.7) over the  $(u, x_4)$  plane gives

$$\int_D \tilde{F}_2 = \theta + i \int_{-L/2}^{L/2} dx_4 \text{tr}(A_u) = \theta + \frac{\sqrt{2N_f}}{f_\pi} \eta', \tag{2.10}$$

where in the second equality we have used a relation between the  $\eta'$  field and the 8-brane gauge field which generalizes the one given in [8] for the special case of antipodal separation between the 8-branes and anti-8-branes at  $u \rightarrow \infty$ . The supergravity action for the RR 1-form is modified,

$$S_{C_1} = \int d^{10}x \sqrt{-g} |\tilde{F}_2|^2, \tag{2.11}$$



**Figure 1:** The 8-brane-anti-8-brane configuration at finite temperature: (a) confining phase (b) deconfinement with broken chiral symmetry (c) deconfinement with restored chiral symmetry.

so the solution with the above condition on the flux is given by

$$\tilde{F}_2 = \frac{c}{u^4} \left( \theta + \frac{\sqrt{2N_f}}{f_\pi} \eta' \right) du \wedge dx_4. \quad (2.12)$$

Plugging back into the action, and using the supergravity result for  $\chi_g$  (2.6), we see that  $\eta'$  has a mass-squared

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2} \chi_g \sim \mathcal{O} \left( \frac{1}{N_c} \right), \quad (2.13)$$

in agreement with the Witten-Veneziano relation.

One comment we would like to add here is that although (2.12) is a solution of the equation of motion  $d * \tilde{F}_2 = 0$ , it does not in general satisfy the modified Bianchi identity that follows from (2.7). This is only satisfied for the constant mode of  $\eta'$ , which is however all we need to compute the mass.

### 2.3 Finite temperature

The finite temperature case was studied in [11]. The finite temperature gauge theory is defined by replacing  $t$  with  $\tau = it$  and imposing the periodicity  $\tau \sim \tau + \beta_\tau$ , where  $\beta_\tau = 1/T$ . That means that the boundary is topologically  $\mathbb{R}^3 \times S_{x_4}^1 \times S_\tau^1 \times S^4$ . There are two possible backgrounds with this boundary. The first is just (2.1) with  $t$  replaced by  $\tau = it$ . The second is given by exchanging  $\tau$  and  $x_4$ :

$$ds^2 = \left( \frac{u}{R} \right)^{3/2} (f(u) d\tau^2 + \delta_{ij} dx^i dx^j + dx_4^2) + \left( \frac{u}{R} \right)^{-3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right). \quad (2.14)$$

In this background

$$f(u) = 1 - \frac{u_T^3}{u^3}, \quad (2.15)$$

where  $u_T$  is related to the periodicity of  $\tau$

$$\beta_\tau = \frac{4\pi}{3} \left( \frac{R^3}{u_T} \right)^{1/2}. \quad (2.16)$$

In this case the  $x_4$  circle remains finite and the  $\tau$  circle shrinks to zero size at  $u = u_T$ . By comparing their free energies one can show that the first background is dominant at low temperatures, and the second background dominates at high temperatures. The transition occurs at  $T = 1/\beta_4$ . In the gauge theory this is a confinement/deconfinement phase transition. The background (2.1) describes the low-temperature confined phase, and the background (2.14) corresponds to the high-temperature deconfined phase.

With the flavor 8-branes and anti-8-branes the model exhibits three phases in general. At low temperatures the background is (2.1) and the 8-branes and anti-8-branes are necessarily in the connected configuration, so both the gluons and the quarks are confined, and chiral symmetry is broken (figure 1a). At high temperatures the background becomes (2.14) and the gluons are deconfined. In this case both the connected and disconnected 8-brane-anti-8-brane configurations are solutions, and one must compare their free-energies to determine which is dominant. At high enough temperatures the disconnected configuration wins and chiral symmetry is restored (figure 1c). However for  $L/R$  below a critical value ( $\sim 0.97$ ) there is an intermediate phase at  $1/\beta_4 < T < 0.154/L$ , in which the 8-branes and anti-8-branes remain connected. In this phase chiral symmetry is broken even though the gluons are deconfined (figure 1b).

### 3. Theta dependence and $\eta'$ mass in the deconfined phase

Let us begin by repeating the computations of the topological susceptibility and the  $\eta'$  mass for finite temperature. Below the deconfinement transition both the background and the 8-brane configuration are essentially unaltered from the zero temperature case. Therefore  $\chi_g$  remains  $\mathcal{O}(1)$  and  $m_{\eta'}^2$  remains  $\mathcal{O}(1/N_c)$  in the confined phase.

In the deconfined phase the topology of the background changes: the  $(u, x_4)$  plane is a cylinder rather than a disk. Therefore in the pure YM case we must replace the condition on the RR flux (2.4) with

$$\int_C F_2 = \theta - \int_{S_{u_T}^1} C_1. \tag{3.1}$$

Now the minimal energy solution to the equation of motion is  $F_2 = 0$ , which can be achieved by adjusting the second term on the right to equal  $\theta$ . This shows that  $\chi_g = 0$  to order 1 in the deconfined phase. Furthermore, since this argument relies only on the topology of the background it is likely to hold to all orders in  $1/N_c$ . This agrees with the expectation that in the absence of confinement the fluctuations of the topological charge density should be suppressed, as can be seen for example in lattice simulations [18]. Including the flavor 8-branes, then in the intermediate phase where the  $\eta'$  exists we must replace the condition on the shifted RR flux (2.10) with

$$\int_C \tilde{F}_2 = \theta - \int_{S_{u_T}^1} C_1 + \frac{\sqrt{2N_f}}{f_\pi} \eta', \tag{3.2}$$

and therefore the minimal energy solution is  $\tilde{F}_2 = 0$ . (The same comment that was made at the end of section 2.2 applies here, namely that this solution satisfies the modified Bianchi



identity only for the constant mode of  $\eta'$ , which is sufficient for computing the mass.) We find that  $m_{\eta'}^2 = 0$ , again to all orders in  $1/N_c$ . As we will now see the  $U(1)_A$  breaking and  $\eta'$  mass in the deconfined phase are due to instantons, precisely as originally proposed by 't Hooft.

## 4. Holographic description of $U(1)_A$ breaking

### 4.1 0-branes as instantons

Yang-Mills instantons correspond to Euclidean D0-branes whose worldlines wind around  $x_4$  [7, 19]. These spacelike D0-branes couple to the  $x_4$  component of  $C_1$  which defines  $\theta$ . For a single instanton, or a small number (relative to  $N_c$ ) of instantons, we can treat the D0-branes as probes in the D4-brane background. The position of the D0-brane in  $\mathbb{R}^4$  corresponds to the position of the instanton, and the position of the D0-brane in the radial coordinate  $u$  is related holographically to the size of the instanton  $\rho$  as

$$\rho^2 = \frac{\beta_4 \lambda}{u}. \tag{4.1}$$

In the low-temperature background (2.1) the D0-brane is unstable (figure 2a) [19]. It can unwind since the circle is topologically trivial. Furthermore, the D0-brane action in this background is given by

$$S_{D0} = m_{D0} \int_0^{\beta_4} dx_4 e^{-\Phi} \sqrt{g_{44}} = \frac{\beta_4}{g_s} \sqrt{1 - \frac{u_\Lambda^3}{u^3}}. \tag{4.2}$$

There is a  $u$ -dependent potential that pulls the D0-brane towards the tip at  $u_\Lambda$ . The instanton therefore grows and disappears at  $\rho \sim \beta_4$ . This is consistent with the large fluctuations of the topological charge density, and confirms Witten's assertion that topological charge is not quantized in the confining theory.

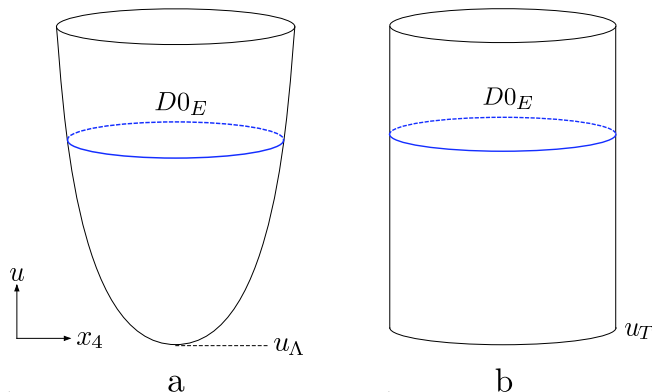
In the high temperature background dual to the deconfined phase (2.14) the  $x_4$  circle is topologically non-trivial, and the D0-brane is stable (figure 2b). Its winding number corresponds to the instanton number. This is consistent with the suppression of the topological charge density fluctuations in this phase. In this background the metric factor exactly cancels the dilaton factor and the action is independent of  $u$ ,

$$S_{D0} = \frac{\beta_4}{g_s} = \frac{8\pi^2 N_c}{\lambda}. \tag{4.3}$$

The instanton size  $\rho$  is therefore a modulus in this phase. The smallest radial position of the D0-brane is  $u_T$ , so instantons have a maximal size  $\sim \beta_\tau$ .

### 4.2 Flavor branes and fermionic zero modes

We now want to consider the effect of a YM instanton at large  $N_c$  with  $N_f$  flavors of massless quarks. In addition to the Euclidean D0-brane on  $x^4$  we have the  $N_f$  D8-branes and  $N_f$  anti-D8-branes which are located at  $x^4 = \mp L/2$ , respectively. The corresponding instanton will have additional degrees of freedom from the  $0_{E-8}$  and  $0_{E-\bar{8}}$  strings. We will



**Figure 2:** Holographic instantons in YM theory: (a) confined phase (b) deconfined phase

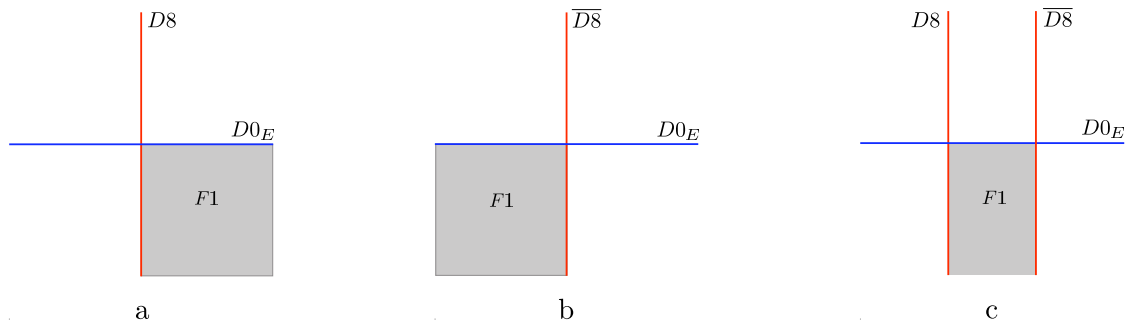
now show that these give the correct fermionic zero modes of the instanton in the presence of the massless quarks.

The  $0_E$ -8 string is an example of an  $ND = 10$  string, i.e. it has mixed boundary conditions in all directions. The bosonic oscillators are all half-odd-integer moded, and the fermionic oscillators are all integer moded in the NS sector, and half-odd-integer moded in the R sector. Thus the roles of the NS and R sector fermions are reversed relative to the  $ND = 0$  case of identical parallel D-branes. In particular, before the GSO projection, the spacetime-bosonic NS ground state is 32-fold degenerate, and the spacetime-fermionic R ground state is non-degenerate. The former is massive, and the latter, as usual, is massless. The action of  $(-1)^F$  is the same as in the  $ND = 0$  case, but with R and NS reversed:

$$\begin{aligned} (-1)^F |0\rangle_R &= -|0\rangle_R \\ (-1)^F |\mathbf{s}\rangle_{NS} &= |\mathbf{s}'\rangle_{NS} \Gamma_{\mathbf{s}'\mathbf{s}}. \end{aligned} \tag{4.4}$$

The GSO projection  $\frac{1}{2}(1+(-1)^F)$  therefore removes the massless fermion from the spectrum of the  $0_E$ -8 string (just like it removed the tachyon from the spectrum of the  $ND = 0$  string). On the other hand if we replace one of the two branes with an antibrane this reverses the GSO projection to  $\frac{1}{2}(1 - (-1)^F)$ , and the massless fermion remains (like the tachyon in the brane-antibrane system). So the massless spectrum of this string includes a single complex fermion degree of freedom for  $0_E\bar{8}$  and  $\bar{0}_E$ -8, and none for  $0_E$ -8 and  $\bar{0}_E\bar{8}$ . In the D4-brane background this gives the correct fermionic zero modes for the instanton and anti-instanton (according to the Atiyah-Singer index theorem). In the instanton case the zero mode is associated with the anti-8-brane, and therefore corresponds to an on-shell left-handed fermion in four dimensions. In the anti-instanton case the zero mode comes from the 8-brane, and therefore corresponds to an on-shell right-handed fermion in four dimensions. For  $N_f$  8-branes we get the required multiplicity of  $N_f$  fermionic zero modes. Note that in the supergravity picture the fermionic zero modes are separate from the quarks. The latter come from the 4-8 and 4- $\bar{8}$  strings and the former from the  $0_E\bar{8}$  or  $\bar{0}_E$ -8 strings.

The presence of the fermionic zero modes in the gauge theory implies a non-trivial multi-fermion condensate which breaks the axial symmetry. How can we interpret this



**Figure 3:** The correct  $D0_E$ -D8-string configurations.

condensate in the dual supergravity picture? As we will now explain the condensate  $\langle \det(\bar{\psi}_R \psi_L) \rangle$  has a very simple geometrical description in the dual background: it is a state of spacelike strings which are enclosed by the 0-brane, 8-branes and anti-8-branes.

### 4.3 Spacelike strings

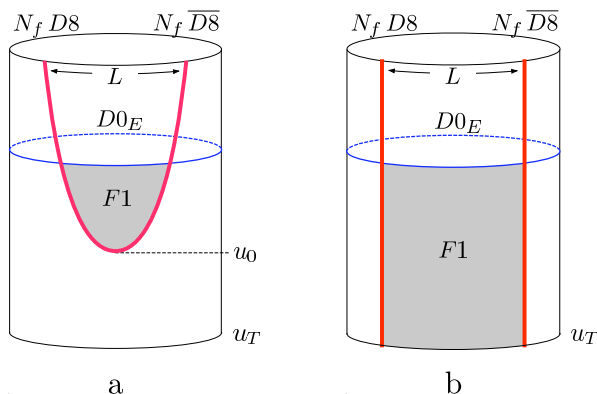
Consider first the  $0_E$ -8 configuration in flat space. As it stands this configuration is incomplete: it requires a semi-infinite spacelike string worldsheet ending on the 0-brane worldline on one side of the 8-brane. This can roughly be thought of as a "superluminal" version of the 0-8 string creation effect [20–23]. It can be proven by an argument similar to the one given in [24] for the ordinary 0-8 configuration. The 8-brane is a source for the RR 0-form field strength  $F_0$  (the Poincare dual of the 10-form  $F_{10}$ ), such that its value jumps by 1 across the 8-brane. We adopt the convention that  $F_0 = 0$  to the left of the 8-brane and  $F_0 = 1$  to the right (this can be achieved by placing another 8-brane at infinity on the left). For the anti-8-brane the left and right sides are reversed. A non-zero  $F_0$  is necessarily a background of massive Type IIA supergravity [25], in which the (sourceless) equation of motion for the  $B$  field is

$$d(*H) = F_0 * F_2. \quad (4.5)$$

Now consider a spacelike 0-brane along  $x_4$  in this background, and integrate the above equation over an  $S^8$  surrounding the 0-brane in Euclidean spacetime. The l.h.s. vanishes, whereas the r.h.s. gives  $F_0$ . We therefore need to include a source term of strength  $F_0$  on the r.h.s., such that its integral on the  $S^8$  cancels this. Geometrically this is a spacelike string worldsheet which is wedged between the 0-brane and the 8-brane on the  $F_0 = 1$  side (figure 3a). For the anti-8-brane the worldsheet is on the other side (figure 3b), and therefore in an  $8\bar{8}$  configuration the worldsheet is enclosed between them (figure 3c). In the appendix we give an alternative worldsheet argument for the existence of this string.

### 4.4 Breaking $U(1)_A$ holographically

Applying this reasoning to the situation at hand we see that the configuration must include a spacelike string for each 8-brane-anti-8-brane pair, which, for the lowest energy, fills the region in the  $(u, x_4)$  plane bounded by the 0-brane and the 8-brane-anti-8-brane pair as



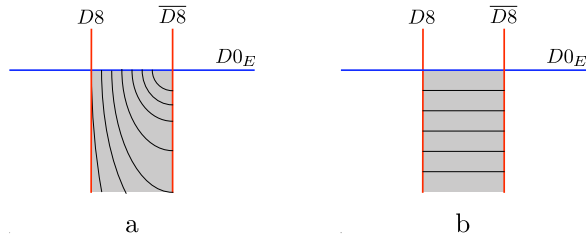
**Figure 4:** Instanton and spacelike strings: (a) broken chiral symmetry phase, (b) restored chiral symmetry phase.

shown in figure 4. We will now argue that these  $N_f$  spacelike strings are the holographic dual of the axial symmetry breaking condensate  $\langle \det(\bar{\psi}_R \psi_L) \rangle$ . To this end we will determine the  $U(N_f)_R \times U(N_f)_L$  quantum numbers of the spacelike string state, and show that they agree with those of the condensate.

The enclosed spacelike string can be quantized using two different parameterizations. Consider first the parameterization where one endpoint of the string is on the 0-brane boundary, and the other endpoint is on the anti-8-brane boundary (in the connected configuration this means on the right half of the 8-brane curve). In this parametrization the worldsheet is foliated by  $0_E - \bar{8}$  strings (figure 5a). The ground state of this string is a single massless fermion. Therefore for  $N_f$   $\bar{8}$ -branes (and  $N_f$  8-branes) the state must be anti-symmetrized in the anti-8-brane Chan-Paton index. Now consider a different parameterization of the worldsheet where one endpoint is on the 8-brane boundary and the other endpoint is on the anti-8-brane boundary (figure 5b). In this case the ground state of the open string is the tachyon. For  $N_f$  8-branes and anti-8-branes there are  $N_f^2$  tachyons  $T^{ij}$ . As we saw in the first parameterization the state must be anti-symmetric in one of the indices, in other words it has the form  $\epsilon_{j_1 \dots j_{N_f}} T^{i_1 j_1} \dots T^{i_{N_f} j_{N_f}}$ . Since  $T^{ij}$  is bosonic the other index is also anti-symmetrized and we get  $\epsilon_{i_1 \dots i_{N_f}} \epsilon_{j_1 \dots j_{N_f}} T^{i_1 j_1} \dots T^{i_{N_f} j_{N_f}} = \det T$ . This has precisely the quantum numbers of the fermionic condensate, so we identify it as the holographic dual,

$$\det T \leftrightarrow \langle \det(\bar{\psi}_R \psi_L) \rangle. \tag{4.6}$$

This identification is also consistent with the identification of the tachyon  $T^{ij}$  as the dual to the fermion bilinear  $\langle \bar{\psi}_R^i \psi_L^j \rangle$ , which is responsible for breaking the chiral symmetry [26]. Note that since the left- and right-handed fermions live at different positions in  $x^4$  the operator on the r.h.s. is gauge invariant only after dimensional reduction to four dimensions. The relation should therefore be understood only in the four-dimensional sense. We conclude that the anti-symmetrized state of spacelike strings is what is responsible for breaking  $U(1)_A$  in the dual supergravity picture.



**Figure 5:** Parameterizations of the spacelike string: (a) foliating with  $D0_E$ - $\overline{D8}$  strings, (b) foliating with  $D8$ - $\overline{D8}$  strings.

#### 4.5 IR regulated instanton gas?

In computing the instanton contribution to the fermionic condensate one has to integrate over the instanton moduli space, including the instanton size  $\rho$ . This integral is IR divergent. Furthermore, in summing over the contributions of multiple instantons and anti-instantons one assumes a dilute gas approximation, which breaks down when the instantons become too large. Finite temperature provides a natural IR cutoff on the instanton size integral. We can see this explicitly in the dual background figure 2b, where there is a minimal value of  $u$  for the 0-brane position. However this will not by itself solve the second problem, since a gas of instantons of size on the order of the cutoff will not be dilute. A possible solution to this problem can be seen in the intermediate phase (figure 4a), which is of-course the one relevant for the  $\eta'$  mass computation.

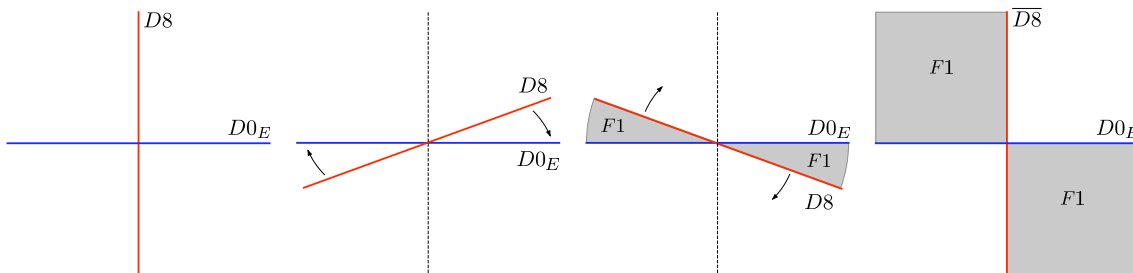
In this phase the 8-branes and anti-8-branes connect at  $u = u_0 \geq u_T$  and the 0-brane crosses them only for  $u > u_0$ . For 0-branes below  $u_0$  the fermionic zero modes are lifted and there is no spacelike string. This is qualitatively what one expects in the gauge theory, since  $u_0$  is the energy scale of chiral symmetry breaking. Above this scale there are free massless quarks and chiral symmetry is restored, so the instanton should have  $N_f$  fermionic zero modes. Below this scale chiral symmetry is broken and quarks and anti-quarks bind into mesons, so the fermionic zero modes should be lifted. Therefore instantons larger than  $\rho_c = \sqrt{\beta_4 \lambda / u_0}$  do not contribute to  $U(1)_A$  breaking. Since this is smaller than the maximal size  $\rho_m = \sqrt{\beta_4 \lambda / u_T}$  the dilute gas approximation might actually be valid in this phase. If so, it would imply an  $\eta'$  mass  $m_{\eta'}^2 \sim \mathcal{O}(e^{-N_c})$  in the intermediate phase.

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#### A. A worldsheet argument for the spacelike string

Another way to deduce the existence of the spacelike string in the  $0_E$ -8 configuration is by computing the vacuum annulus diagram of the  $0_E$ -8 string. The analogous computation



**Figure 6:** The creation of a Euclidean string worldsheet in a  $D0_E$ - $D8$  system.

in the 0-8 case was done in [27, 23]. In the present case the string has  $ND = 10$ , which means all the bosonic oscillators are half-odd-integer moded, and the fermionic oscillators are all integer moded in the NS sector and all half-odd-integer moded in the R sector. The R ground state is therefore non-degenerate (and massless). The ghosts and superghosts have the usual moding. Defining as usual  $q = e^{-\pi t}$  and

$$f_1(q) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) \quad (\text{A.1})$$

$$f_2(q) = \sqrt{2} q^{1/12} \prod_{n=1}^{\infty} (1 + q^{2n}) \quad (\text{A.2})$$

$$f_3(q) = q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}) \quad (\text{A.3})$$

$$f_4(q) = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}), \quad (\text{A.4})$$

the vacuum annulus amplitude can be written as

$$A_{0_E-8} = \int_0^{\infty} \frac{dt}{2t} \frac{f_1^2}{f_4^{10}} \left[ -\frac{f_3^{10}}{f_2^2} + \frac{f_2^{10}}{f_3^2} \pm \frac{f_4^{10}}{f_1^2} \cdot \infty \right]. \quad (\text{A.5})$$

The sign in the third term is associated with the sign in the GSO projection (in the R sector), and therefore depends on whether we have an 8-brane or an anti-8-brane, and the infinite factor comes from the trace over the superghost zero modes. We can now think of a process where we rotate the 8-brane such that it crosses the  $0_E$ -brane. Once rotated by  $\pi$  the 8-brane becomes an anti-8-brane, so the sign in (A.5) flips. The difference between the initial and final amplitudes is infinite, and is accounted for by the creation of a spacelike string (figure 6). In fact this is just the ordinary string creation effect: when the 8-brane crosses the  $0_E$ -brane they are parallel, and the configuration is equivalent to the 0-8 system. The divergence in this case is associated with the infinite extent of the spacelike worldsheet of the string. To make the picture symmetric between the 8-brane and anti-8-brane cases we need to start with a worldsheet in the lower-right quadrant for the 8-brane, which has the opposite orientation of the one which is created in the crossing. The configuration after the rotation will have an anti-8-brane with a worldsheet in the lower-left quadrant (figure 3).

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